Lab 11

# LRC Circuits, Damped Forced Harmonic Motion

## What You Need To Know:

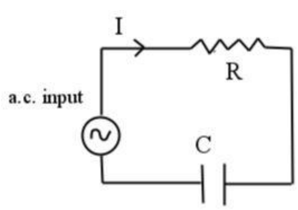
**The Physics** OK this is basically a recap of what you've done so far with RC circuits and RL circuits. Now we get to put everything together and treat the full LRC circuit.

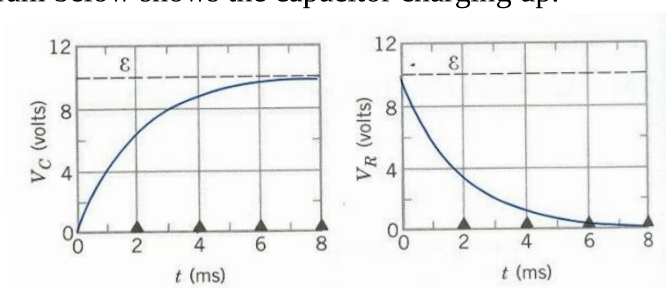
Let's recap!

**RC circuits:**

In a previous lab we found that for a charging capacitor, the voltage loop law gives, $\varepsilon - IR - \frac{q}{C} = 0$. Here the ac supply has voltage ε. In terms of the charge we may write, $C\varepsilon = RC\frac{dq}{dt} + q$.

We may solve this to find, $q(t) = C\varepsilon\left(1 - e^{-t/\tau_C}\right)$. Then $I(t) = \frac{\varepsilon}{R}e^{-t/\tau_C}$. The time constant is $\tau_C = RC$. The diagram below shows the capacitor charging up.

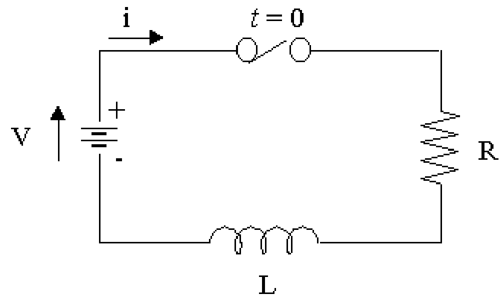


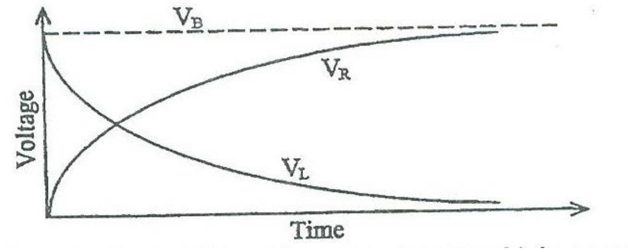




You find $t_{1/2}$ from looking at the either the voltage gain across the capacitor or voltage drop over the resistor. When the voltage is half the max or min value measure $t_{1/2}$ from the time scale, using the relevant time/div. scaling on the scope. You can use $t_{1/2}$ to find the time constant $\tau_C$ . A very similar situation arises with RL circuits.

**RL circuits:**

When you have a battery (or signal generator) with voltage, ε, and connect a resistance R and inductance L to it, the voltage loop equation gives us, $\varepsilon - L\frac{dI}{dt} - IR = 0$, which can be solved for current I(t) as, $I(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau_L}\right)$, here the time constant is $\tau_L = \frac{L}{R}$. $(V_B = \varepsilon)$

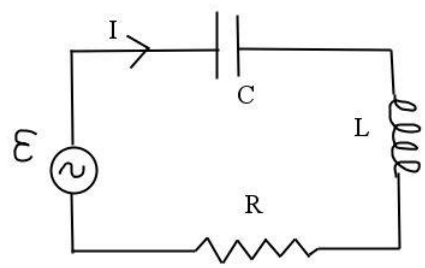



You find $t_{1/2}$ from looking at the either the voltage drop across the inductor or voltage build up on the resistor. When the voltage is half the max or min value measure $t_{1/2}$ from the time scale, using the relevant time/div. scaling on the scope. So finally we come to LRC circuits…

**LRC circuit:**

This is a perfect example of a damped driven harmonic oscillator which you will find throughout your studies in physics in mechanics as well as optics and electronics. You can Google it and find demo's online. Check out for example http://en.wikipedia.org/wiki/RLC_circuit .




Instead of a direct ac supply we will be using a mutual inductance from a primary coil. The coil shown here is the secondary. We will connect the primary to a square wave signal generator. We've done this before in the last experiment Lab 10.

Look at the voltage drops around the LRC circuit, $\varepsilon = IR + L\frac{dI}{dt} + \frac{q}{C}$. Divide this by $L$ and using $I = \frac{dq}{dt}$ , we get, $\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{\varepsilon}{L}$.

This second order differential equation can be solved by standard methods, complementary function etc, to find, $q(t) = q_o e^{-Rt/2L}\cos(\omega t + \phi)$. From which we find the voltage across the capacitor as $V_C(t) = \frac{q(t)}{C}$ where $\tan\phi = \frac{X_L - X_C}{R}$ and $\omega = \sqrt{\left(\frac{R}{2L}\right)^2 - \omega_o{}^2}$ where $\omega_o = \frac{1}{\sqrt{LC}}$ is the natural or resonance frequency of the oscillator. We define the impedance as $Z = \sqrt{R^2 + (X_L - X_C)^2}$ . Then we can write the supply max voltage $\varepsilon_o = I_o Z$ . You can find the analysis online and in Halliday and Resnick "Physics" Vol II. The impedance Z is a complex quantity, $Z = R + i(X_L - X_C)$ you can find the magnitude and phase in the usual way (on the next page).

$$|Z| = \sqrt{R^2 + (X_L - X_C)^2} \text{ And } Z = |Z|e^{i\phi} \text{ where } \phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Note that resistance R is the real part of impedance and the complex part is the reactance $(X_L - X_C)$. When energy flows around a circuit, the resistance will cause energy loss as heat dissipation, the reactance stores the energy in either the electric field (for the capacitor) or the magnetic field (for the inductor). This energy can be released when the capacitor discharges or the solenoid (inductor) powers down. So with resistance you lose energy, with reactance you can get the energy back.

*Maybe you don't want to involve second order differential equations. OK let's solve the problem another way.*

- Let's look at the voltage drop across the resistor:

  $V_R = IR = I_o R\sin(\omega t - \phi)$ Taking $I(t) = I_o \sin(\omega t - \phi)$.

  Note that the voltage across the resistor is in phase with the current.

- Let's look only at the voltage drop across the capacitor:

  $$V_C = \frac{q}{C} = \frac{1}{C}\int I\,dt = -\frac{I_o}{\omega C}\cos(\omega t - \phi)$$

  $$V_C = \frac{I_o}{\omega C}\sin\left(\omega t - \phi - \frac{\pi}{2}\right)$$

  Here you can see that the current reaches a maximum ¼ cycle (or π/2) ahead of $V_C$.

  **Current leads voltage by π/2 in a capacitor.**

  We define the capacitive reactance as $X_C = \frac{1}{\omega C}$ then we get,

  $$V_C = I_o X_C \sin\left(\omega t - \phi - \frac{\pi}{2}\right).$$

- Let's look at the voltage drop across the inductor:

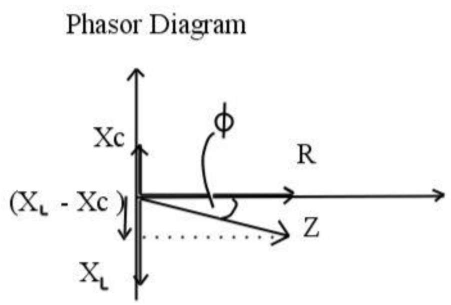
  $$V_L = L\frac{dI}{dt} = LI_o\omega\sin\left(\omega t - \phi + \frac{\pi}{2}\right)$$

  Here we can see that the voltage across the inductor is π/2 out of phase with the current and it lags behind the current by π/2. Take the inductive reactance to be $X_L = \omega L$, then the voltage across the inductor is,

  $V_L = X_L I_o \sin\left(\omega t - \phi + \frac{\pi}{2}\right)$. **Voltage leads current by π/2 in an inductor.**

**As a memory aid, remember "ELI the ICE man".**

- In ELI, the E is the voltage, L is the inductor, and I is the current. So, for an inductor, L, the voltage, E, leads the current, I, since E comes before I in ELI.
- In ICE, the I is the current, C is the capacitor, and E is the voltage. So, for a capacitor, C, the current, I, leads the voltage, E, since I comes before E in ICE.




$$\tan\phi = \frac{X_L - X_C}{R}$$

Ok let's tie everything together now… the solution without solving a differential equation follows:

Voltage drop around the LRC circuit is:

$$\varepsilon - V_R - V_L - V_C = 0$$

Use $\varepsilon = \varepsilon_o \sin(\omega t)$, then we get;

$$\varepsilon_o \sin(\omega t) = I_o R \sin(\omega t - \phi) + I_o X_L \sin\left(\omega t - \phi + \frac{\pi}{2}\right) + I_o X_C \sin\left(\omega t - \phi - \frac{\pi}{2}\right)$$

$$\varepsilon_o \sin(\omega t) = I_o\{R\sin(\omega t - \phi) + (X_L - X_C)\cos(\omega t - \phi)\}$$

This can be solved via trig methods to give…

$$\varepsilon_o \sin(\omega t) = I_o\sqrt{R^2 + (X_L - X_C)^2}\sin(\omega t)$$

Where the impedance is given by $Z = \sqrt{R^2 + (X_L - X_C)^2}$ hence simply put the max voltage of the supply, $\varepsilon_o = I_o Z$ you can cancel the sine terms on both sides!

Also $\omega = 2\pi f$ where $f$ is the frequency in Hz or per second. The frequency f = *1/T* where *T* is the period of the oscillation. The impedance is a minimum when $X_L = X_C$. Then the natural frequency is given by $\omega = \omega_o = 1/\sqrt{LC}$.

The energy in the circuit sloshes back and forth between the capacitor and the inductor… the oscillations are damped out by the resistance in the circuit. The capacitor charges when the coil powers down, then the capacitor discharges and the coil powers up… and so on. There is a natural frequency to this oscillation… you should try to determine for your circuit. With no resistance it would be the natural frequency $\omega 0$ described above.

## What You Need To Do:

## Experiment 1

**A)** Take the secondary coil of 2000 turns and hook it up to the multi-meter to determine its resistance. Write that down.

**B)** Get the 0.05 μF capacitor (lower voltage rating will be fine) and stick it into the circuit board.

**C)** Set the signal generator to square wave pulse, about 60 Hz (use the 100 Hz range for this).

**D)** Hook up the primary coil 400 turns to the signal generator turn on the signal. Slide the plastic rod through both coils to hold them in place and have them touching… some tape might be used to hold them together throughout the experiments.

**E)** Connect the secondary coil in series with the capacitor. Then attach the scope CH 1, to the +/– end of the capacitor. *See Fig 1. For setup.*

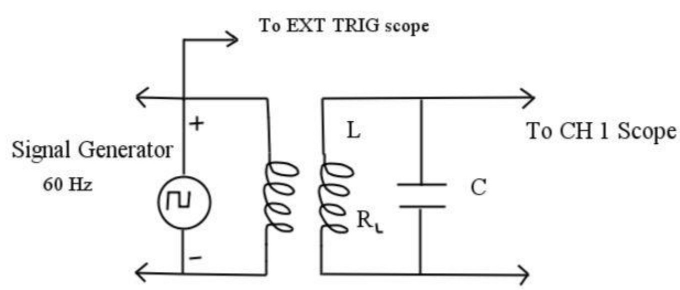
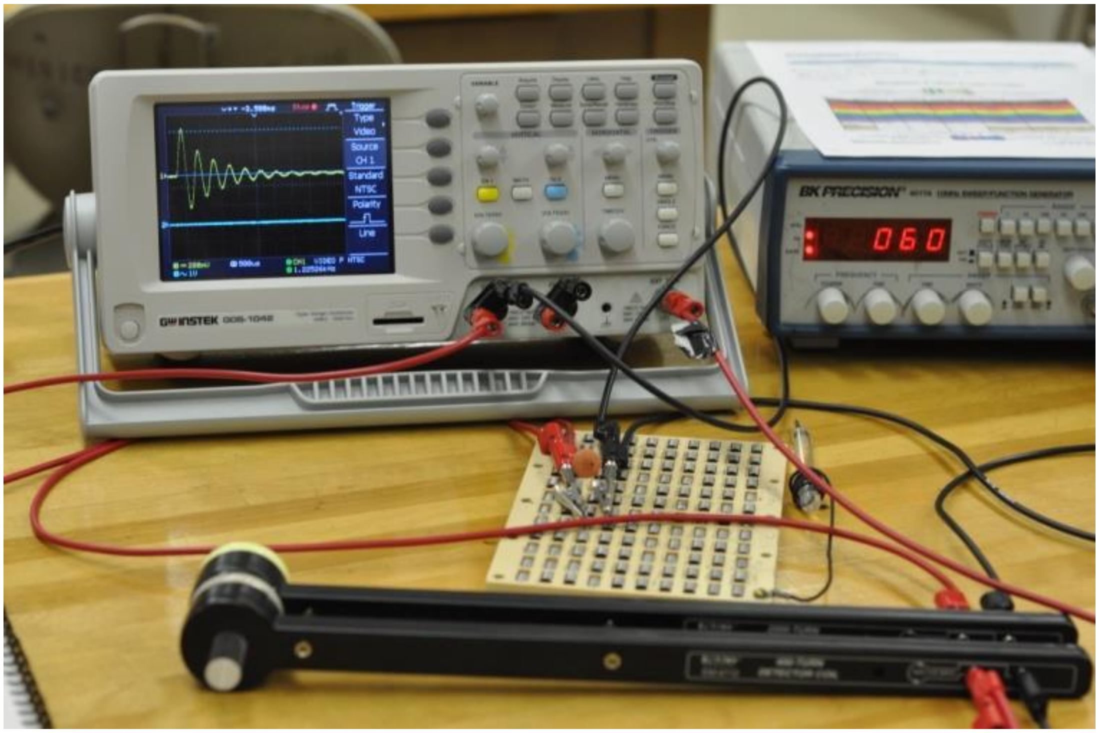



Fig 1.

**F)** Connect the positive end of the primary coil to the EXT trigger source of the scope. We will be using the ext trigger for this experiment.

**G)** Turn on the oscilloscope. Set volts/div CH 1 to 200mV and time/div to 500 μs. Choose menu, ext trigger source for CH 1.

With everything switched on you should be seeing a damped oscillatory curve like the one in the photo below.




If necessary press the run/stop button and use the horizontal shift knob to get the full damped curve in view.

Note the red lead on the right bottom of the scope is the Ext trigger. This is going to the positive end of the primary coil, which in turn is connected to the signal generator.

$R_L$ is the resistance of the secondary coil (inductor) which you measured with the multi-meter ~ 250Ω.

You should see nice damped oscillations as shown above. Using cursors record Voltage and time for each peak of the signal. Plot the data in excel and fit an exponent to your curve. Recall this is damped harmonics from equation $q(t) = q_o e^{-Rt/2L} \cos(\omega t + \phi)$

(Note: Since the signal is oscillating only collect the magnitude of the peaks)

**1)** Find the time constant for the decay τ?

**2)** What is $t_{1/2}$? Calculate L for the inductor.

**3)** Natural oscillation of the circuit ω?

**4)** Calculate L using ω Compare this value to your calculation from part 2.

## Experiment 2

Now hook up the resistance box in series with the secondary coil as shown below. You need to see what happens when you add in extra external resistance in series with the resistance from the secondary coil. This should damp out the oscillations faster and produce over damping.

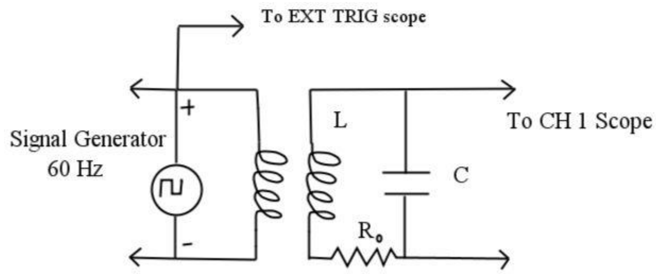
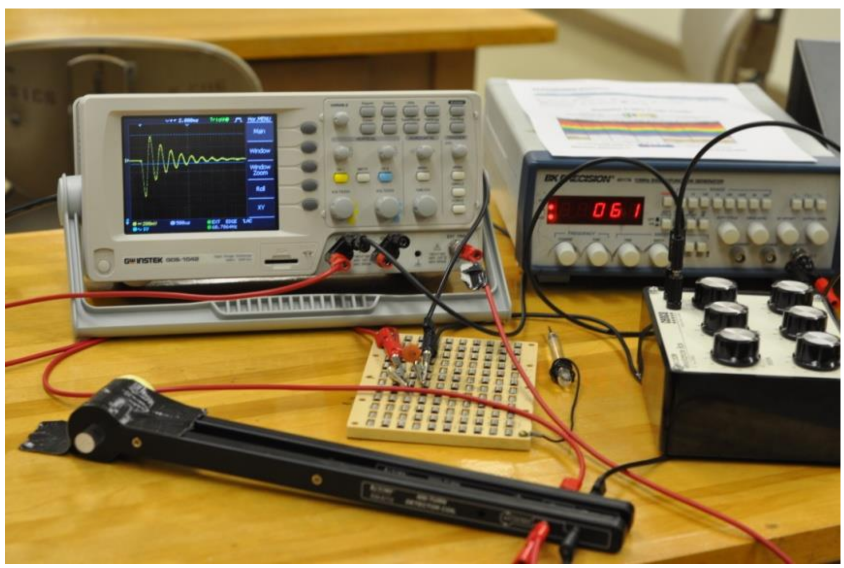



Fig 2.

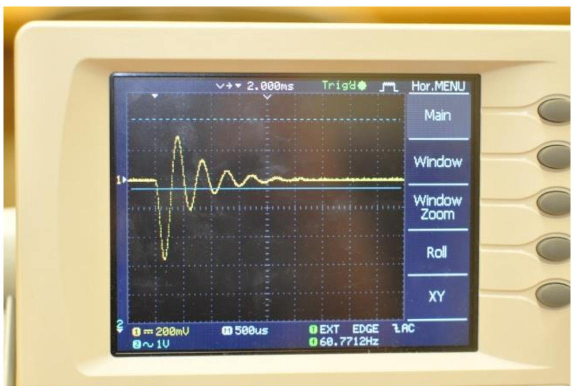


What follows is a sequence of photos showing how the oscillations are gradually damped by adding larger resistors to the circuit. Eventually you get over-damping, which is the case where you hardly see any oscillation at all. The full resistance in the circuit is R = R0 + RL where RL is the coil (inductor) resistance which you can measure using the meter provided before you hook up the coil into the circuit.

**$R_L$ is approx 250 Ω for all.**

Make sure to notice how the oscillations are damped out by adding resistance!


$R_0 = 100\ \Omega$ from resistance box.




$R_0 = 300\ \Omega$


$R_0 = 600\ \Omega$

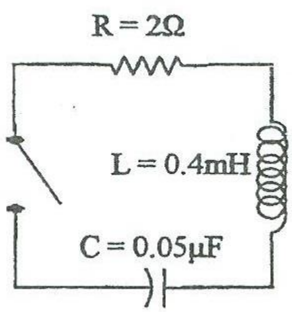

$R_0 = 1000\ \Omega$ near critical damping.

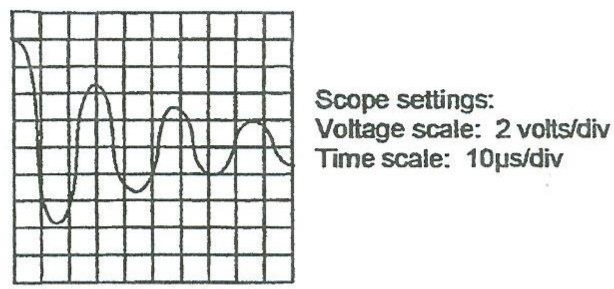
Over-damping has no oscillation at all just exponential decay.

Look up under damped, critically damped and over damped oscillation online!

## POST LAB EXERCISE

Consider the following circuit consisting of a capacitor C = 0.05 μF, and a coil of inductance L = 0.4 mH and internal resistance $R_L$ = 2.0 Ω. The capacitor is charged to V = 10 volts and then the switch is closed. The voltage across the capacitor as displayed on the oscilloscope is shown below.




Predict the expected values of:

**a)** Free oscillation period T and frequency $\omega_0 = 2\pi f$.

**b)** Time constant for the decay, $\tau$. Use the half life method you have done before $\tau = t_{1/2} / \ln 2$ .

***HINT:*** *Note that the voltage across the capacitor is expected to vary as;*

$$V_C(t) = \frac{q(t)}{C} = \varepsilon_o e^{-t/\tau} \cos(2\pi f t) = I_o Z e^{-t/\tau} \cos(\omega t)$$

*Successive peaks occur approximately when the cosine becomes unity and are roughly at intervals at period T. A plot of the curve through the successive peaks is an approximation to the curve* $\varepsilon_o e^{-t/\tau}$ .

*(A more accurate result for the period would be gained by differentiating $V_c$ and setting that equal to zero... find where $V_c$ becomes zero… find the freq or period from that.)*